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Computation of the Improvement of Sensitivity of Search Analyses with Increased Integrated Luminosity

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Abstract

The improvement of sensitivity expressed in a limit on $\sigma \times \mathrm{BR}$ for some process is shown to scale faster than the increase in integrated luminosity. This demonstration is done under the assumptions that there are that no events and no background observed, and that the systematic uncertainty in the background has a statistical component that scales with simple Gaussian statistics and a systematic component that is independent of the integrated luminosity.

Contents

1 Introduction

In this computation we show how the limit on a signal depends on the statistcal uncertainty and on the systematic uncertainty which has a component that depends on statistics plus one that is independent of statistics.

First, we show that a limit on $\sigma \times \mathrm{BR}$ for some process improves with a factor of f integrated luminosity as:

$$\sigma \times \mathrm{BR} = \frac{\sigma \times \mathrm{BR}_0^0}{f} \left(1 + \frac{\Delta B_0}{N_s} \frac{1}{\sqrt{f}} + \frac{\Delta S_0}{N_s} \right) \tag{1}$$

where:

$\sigma \times \mathrm{BR}_0^0$	=	statistical poisson limit assuming no events observed
ΔB_0	=	statistical uncertainty in background
ΔS_0	=	systematic uncertainty in background
N_s	=	Number of signal events

For simplicity we assume that no signal nor background events are observed.

Next, we consider the effective gain in the limit with integrated luminosity defined as:

$$\frac{\sigma \times \mathrm{BR}'}{\sigma \times \mathrm{BR}_0} = \frac{1}{f'} \tag{2}$$

where f' is the apparent gain in the integrated luminosity and is show to be

$$f' = f \frac{(1+a)}{(1 + \frac{a}{\sqrt{f}})} \tag{3}$$

where

$$a = \frac{\Delta B_0}{(N_s + \Delta S_0)}. \tag{4}$$

We show then that $f' > f$ and has the limit $f' = f(1+a)$ for large statistical gain.

In order to demonstrate this, we consider an example of setting a 90% CL in 360pb^{-1} of integrated luminosity where the background is 0 ± 3 events with a theoretical uncertainty of 0 ± 1 events. The background is assumed to have been measured by some statistically limited method. Given this, we define:

$\sigma \times \mathrm{BR}_0^0$	=	$2.1/360$ pb^{-1}	The 90% CL limit without background, no events observed
	=	5.8 fb	has this numerical value for 360 pb-1
$\Delta B_0/N_s$	=	3/2.1 = 1.43	assume 0 ± 3 background events measured
$\Delta S_0/N_s$	=	$1/2.1$ = 0.48	assume systematic uncerainty of one event

The scaling law is:

$$\sigma \times \mathrm{BR} = \frac{5.8}{f}\left(1.48 + \frac{1.43}{\sqrt{f}}\right) \mathrm{fb} \tag{5}$$

In the sections that follow, we derive these formulas by first considering the limit in the case of no background or systematic error, then with background that is known with some statistical accuracy, and finally with a systematic uncertainty included. The last section examines the effective gain in the limit with increased integrated luminosity.

2 Computation of the perfect limit

Let us express the limits in units of some standard model cross section times branching ratio:

$$R = \frac{\sigma \times \mathrm{BR}_{lim}}{\sigma \times \mathrm{BR}_{SM}} \tag{6}$$

If we consider the example of the Higgs search in the mode H$\rightarrow$W$^+$ W$^-$, our sensitivity is a factor 12 away from the Standard Model, so $R = 12$. If there is no background and no error on this background, then this limit scales with the luminosity so that if we get a factor of three more luminosity, then we have $R = 12/3 = 4$.

However, if there is no measured background, we expect that we would have systematic uncertainty in the background that depends on the statistical test we have on the background. We also may have a background systematic uncertainty that does not depend on statistics but on a theoretical uncertainty that does not improve with more data. We now turn to considering the limits first without any uncertainties and then with the uncertainties included.

In the perfect situation, with no uncertainties, if we see no events, we compute the limit by assuming 2.1 events for a 90% CL.

Suppose integrated luminosity, $L^{int} = 360pb^{-1} \equiv L_0^{int}$. Then the number of events observed is:

$$N = (\sigma \times \mathrm{BR})\, L^{int} \tag{7}$$

$$\begin{aligned}
\sigma \times \mathrm{BR} &= N/L^{int} && (8)\\
&= N/L_0^{int} && (9)\\
&= 2.1/360pb^{-1} && (10)\\
&= 5.8 \times 10^{-3}pb && (11)\\
&= 5.8\mathrm{fb} && (12)\\
&\equiv \sigma \times \mathrm{BR}_0^0, && (13)
\end{aligned}$$

where we use the subscript to indicate initial limit for the starting integrated luminosity, L_0^{int} and the superscript to indicate that this is in the case of no background and no systematic uncertainty.

If we double the luminosity

$$\begin{aligned}
\sigma \times \mathrm{BR} &\equiv \sigma \times \mathrm{BR}' && (14)\\
&= \frac{N}{2L_0^{int}} && (15)\\
&= \frac{1}{2}\frac{N}{L_0^{int}} && (16)\\
&= \frac{1}{2}\sigma \times \mathrm{BR}_0^0 && (17)\\
&= 5.8\mathrm{fb}/2 && (18)\\
&= 2.9\mathrm{fb} && (19)
\end{aligned}$$

We see then that

$$\sigma \times \mathrm{BR}' = \frac{\sigma \times \mathrm{BR}_0^0}{2} \tag{20}$$

or

$$\sigma \times \mathrm{BR}' = \frac{1}{f}\left(\sigma \times \mathrm{BR}_0^0\right) \tag{21}$$

where f is the luminosity factor between the old and new limit.

3 Computation of the Limit with Statistical Background Error

If we see no events but our background estimate is 0 ± 3 events, then our limit should be computed on a basis of $2.1 + 3 = 5.1$ events. We lose about a factor of 2 over the perfect knowledge described in the previous section. Let us define: $B_0 \equiv 0 \pm 3$ and $\Delta B_0 = 3$. Then

$$\begin{aligned}
\sigma \times \mathrm{BR} &= \frac{N_s + \Delta B_0}{L_0^{int}} && (22)\\
&= \frac{5.1}{L_0^{int}} && (23)\\
&= \frac{5.1}{2.1}\frac{2.1}{L_0^{int}} && (24)\\
&= \frac{5.1}{2.1} 5.8\mathrm{fb} && (25)\\
&= 2.4 \times 5.8\mathrm{fb} && (26)\\
&= 14.1\mathrm{fb} && (27)
\end{aligned}$$

or

$$\begin{aligned}
\sigma \times \mathrm{BR} &= \frac{N}{L_0^{int}} + \frac{\Delta B_0}{L_0^{int}} && (28)\\
&= 2.1/L_0^{int} + 3/L_0^{int} && (29)\\
&= \sigma \times \mathrm{BR}_0^0 + \frac{\Delta B_0}{L_0^{int}} && (30)\\
&= \sigma \times \mathrm{BR}_0^0 + \sigma \times \mathrm{BR}_0^B && (31)\\
&= 5.8\mathrm{fb} + 8.3\mathrm{fb} && (32)\\
&= 14.1\mathrm{fb}. && (33)
\end{aligned}$$

If we double the statistics, then $\Delta B = \Delta B_0/\sqrt{2} = \Delta B_0/\sqrt{f}$ where $f = 2$ is the factor of statistics gained. In this case,

$$\begin{aligned}
\sigma \times \mathrm{BR} &= \frac{2.1 + \Delta B}{L^{int}} && (34)\\
&= \frac{2.1 + \frac{\Delta B_0}{\sqrt{f}}}{f L_0^{int}} && (35)\\
&= \frac{2.1}{f L_0^{int}} + \frac{\Delta B_0}{f\sqrt{f} L_0^{int}} && (36)\\
&= \frac{\sigma \times \mathrm{BR}_0}{f} + \frac{\sigma \times \mathrm{BR}_0^{Bkg}}{f\sqrt{f}} && (37)\\
&= \frac{5.8}{2} + \frac{8.3}{2\sqrt{2}} && (38)\\
&= 2.9\mathrm{fb} + 2.9\mathrm{fb} && (39)\\
&= 5.8\mathrm{fb}. && (40)
\end{aligned}$$

4 Computation Based on Statistical Background and Constant Systematic Error

Now let's add a systematic uncertainty that is independent of luminosity.

Suppose we have a systematic uncertainty of 1 event. (One can argue that we should take this as ± 0.5 evts, but let's take the error as 1 event.)

$$\begin{aligned}
\sigma \times \mathrm{BR}_0 &= \frac{N}{L_0^{int}} + \frac{\Delta B_0}{L_0^{int}} + \frac{\Delta S_0}{L_0^{int}} && (41)\\
&= \frac{2.1}{L_0^{int}} + \frac{3}{L_0^{int}} + \frac{1}{L_0^{int}} && (42)\\
&= \sigma \times \mathrm{BR}_0^0 + \sigma \times \mathrm{BR}_0^{Bkg} + \sigma \times \mathrm{BR}_0^{Sys} && (43)\\
&= \sigma \times \mathrm{BR}_0^0 \left(1 + \frac{\sigma \times \mathrm{BR}_0^{Bkg}}{\sigma \times \mathrm{BR}_0^0} + \frac{\sigma \times \mathrm{BR}_0^{Sys}}{\sigma \times \mathrm{BR}_0^0}\right) && (44)\\
&= \sigma \times \mathrm{BR}_0^0 \left(1 + \frac{\Delta B_0}{N_s} + \frac{\Delta S_0}{N_s}\right) && (45)
\end{aligned}$$

where we define ΔB_0 as the error on the background, N_s the number of signal events and ΔS_0 the systematic error.

If we increase the integrated luminosity by a factor of f, this becomes:

$$\begin{aligned}
\sigma \times \mathrm{BR} &= \frac{\sigma \times \mathrm{BR}_0^0}{f} + \frac{\sigma \times \mathrm{BR}_0^{Bkg}}{f\sqrt{f}} + \frac{\sigma \times \mathrm{BR}_0^{Sys}}{f} && (46)\\
&= \frac{\sigma \times \mathrm{BR}_0^0}{f} \left\{1 + \frac{\sigma \times \mathrm{BR}_0^{Bkg}}{\sigma \times \mathrm{BR}_0^0}\frac{1}{\sqrt{f}} + \frac{\sigma \times \mathrm{BR}_0^{Sys}}{\sigma \times \mathrm{BR}_0^0}\right\} && (47)\\
&= \frac{\sigma \times \mathrm{BR}_0^0}{f} \left(1 + \frac{\Delta B_0}{N_s}\frac{1}{\sqrt{f}} + \frac{\Delta S_0}{N_s}\right) && (48)
\end{aligned}$$

Let's check this expression with an example. We start out with:

$$\begin{aligned}
\sigma \times \mathrm{BR}_0 &= \left(\frac{2.1}{360} + \frac{3}{360} + \frac{1}{360}\right) \mathrm{fb} && (49)\\
&= (5.8 + 8.3 + 2.8)\,\mathrm{fb} && (50)\\
&= 16.9\mathrm{fb}. && (51)
\end{aligned}$$

If we double the integrated luminosity, then we have

$$\begin{aligned}
\sigma \times \mathrm{BR} &= \left(\frac{2.1}{2 \times 360} + \frac{\frac{3}{\sqrt{2}}}{2 \times 360} + \frac{1}{2 \times 360}\right) \mathrm{fb} && (52)\\
&= (2.9 + 2.9 + 1.4)\,\mathrm{fb} && (53)\\
&= 7.3\mathrm{fb}. && (54)
\end{aligned}$$

With now compute the quantites needed for the formula in Eqn. 48

$$\begin{aligned}
\sigma \times \mathrm{BR}_0^0 &= 5.8\mathrm{fb} && (55)\\
\frac{\Delta B_0}{N_s} &= \frac{3}{2.1} && (56)\\
&= 1.43 && (57)\\
\frac{\Delta S_0}{N_s} &= \frac{1}{2.1} && (58)\\
&= 0.48 && (59)\\
& && (60)
\end{aligned}$$

so that

$$\begin{aligned}
\sigma \times \mathrm{BR} &= \frac{5.8}{f}\left(1 + \frac{1.43}{\sqrt{f}} + 0.48\right) && (61)\\
&= \frac{5.8}{f}\left(1.48 + \frac{1.43}{\sqrt{f}}\right), && (62)
\end{aligned}$$

which, for $f = 2$ is

$$\begin{aligned}
\sigma \times \mathrm{BR} &= 2.9\mathrm{fb}\,(1.48 + 1.01) && (63)\\
&= 2.9\mathrm{fb} \times 2.49 && (64)\\
&= 7.3\mathrm{fb} && (65)
\end{aligned}$$

5 Computation of the effective gain in luminosity

Now consider that we are interested in comparing how our $\sigma \times \mathrm{BR}$ sensitivity has improved with luminosity. We wish to compute

$$\sigma \times \mathrm{BR}' = \frac{\sigma \times \mathrm{BR}_0}{f'} \quad (66)$$

where the "$'$" indicates quantities after more luminosity and "$_0$" indicates sensitivities with the current integrated luminosity. We can solve for $1/f'$:

$$\frac{1}{f'} = \frac{\sigma \times \mathrm{BR}'}{\sigma \times \mathrm{BR}_0} \quad (67)$$

We then use Eqn. 48 to compute $\sigma \times \mathrm{BR}'$, Eqn. 45 to compute $\sigma \times \mathrm{BR}_0$ and write the ratio:

$$\frac{\sigma \times \mathrm{BR}'}{\sigma \times \mathrm{BR}_0} = \frac{\left[\left(\frac{\sigma \times \mathrm{BR}_0^0}{f}\right)\left(1 + \frac{\Delta B_0}{N_s}\frac{1}{\sqrt{f}} + \frac{\Delta S_0}{N_s}\right)\right]}{\left[\sigma \times \mathrm{BR}_0^0\left(1 + \frac{\Delta B_0}{N_s} + \frac{\Delta S_0}{N_s}\right)\right]} \quad (68)$$

$$= \frac{\left[\frac{1}{f}\left(1+\frac{\Delta S_0}{N_s}+\frac{\Delta B_0}{N_s}\frac{1}{\sqrt{f}}\right)\right]}{\left[\left(1+\frac{\Delta S_0}{N_s}+\frac{\Delta B_0}{N_s}\right)\right]} \tag{69}$$

$$= \frac{1}{f}\frac{\left\{1+\left[\frac{\frac{\Delta B_0}{N_s}}{1+\frac{\Delta S_0}{N_s}}\right]\left[\frac{1}{\sqrt{f}}\right]\right\}}{1+\left[\frac{\frac{\Delta B_0}{N_s}}{1+\frac{\Delta S_0}{N_s}}\right]} \tag{70}$$

where, as described above, ΔB_0, N_s and ΔS_0 are the number of background, signal assumed and systematic. These take on the values $\Delta B_0 = 3$, $N_s = 2.1$ and $\Delta S_0 = 1$ for the example here.

If we let

$$a = \frac{\frac{\Delta B_0}{N_s}}{1+\frac{\Delta S_0}{N_s}} \tag{71}$$

$$= \frac{\Delta B_0}{N_s + \Delta S_0}, \tag{72}$$

we have

$$\frac{\sigma \times \mathrm{BR}'}{\sigma \times \mathrm{BR}^0} = \frac{1}{f}\frac{1+\frac{a}{\sqrt{f}}}{1+a} \tag{73}$$

$$= \frac{1}{f}\frac{1}{\frac{1+a}{1+\frac{a}{\sqrt{f}}}} \tag{74}$$

$$= \frac{1}{f'}. \tag{75}$$

Therefore

$$f' = f\frac{1+a}{1+\frac{a}{\sqrt{f}}}. \tag{76}$$

and hence $f' > f$.

Let's work out the numbers, first by considering what happened after we doubled the luminosity.

In that case we still assume we have no events and take the Poisson 90% CL of 2.1 events. We take $3/\sqrt{2}$ events for background, assuming that we still have no background and that we have gained in our uncertainty of that fact. We still take a systematic uncertainty of 1 event, assuming that all of our gain in understanding of systematics comes in the background statistics.

We have a new limit of

$$\sigma \times \mathrm{BR}' = \left(\frac{2.1}{2\times 360}+\frac{\frac{3}{\sqrt{2}}}{2\times 360}+\frac{1}{2\times 360}\right)\mathrm{fb} \tag{77}$$

$$= (2.92+2.95+1.39)\,\mathrm{fb} \tag{78}$$

$$= 7.26\mathrm{fb} \tag{79}$$

Now, from Eqn. 51, $\sigma \times \mathrm{BR}_0 = 16.9\mathrm{fb}$, so that we have an improvement of 16.9/7.26 = 2.34. This is better than the factor of two in integrated luminosity, so it seems that uncertainty is a good thing! In fact it is not. If we had not had any uncertainty, we would have had a limit of 2.92 fb which is 7.26/2.92 = 2.49 times *worse* due to this background. The point is that we are improving in our understanding of background and in our statistics so we win more than luminosity would suggest but we are always worse that we could be.

If we use the formula to compute f', we have

$$
\begin{aligned}
a &= \frac{\Delta B_0}{N_s + \Delta S_0} && (80)\\
&= \frac{3}{2.1+1} && (81)\\
&= \frac{3}{3.1} && (82)\\
&= 0.97 && (83)
\end{aligned}
$$

For f=2 we have

$$
\begin{aligned}
f' &= f\frac{1+a}{1+\frac{a}{\sqrt{f}}} && (84)\\
&= 1.17f && (85)\\
&= 2.34, && (86)
\end{aligned}
$$

as we have seen. For large statistics, we have $f' = f\,(1+a) = 1.97f$.

That is, we eventually gain about twice the integrated luminosity.

To improve these estimates, we would have to look at integrals of experiments with assumptions of various signal and background levels weighted by Poisson statistics. The point here was to show that we gain at least as fast as the luminosity.

6 Conclusions and Future

We have shown that the gain in a limit on $\sigma \times \mathrm{BR}$ with a factor of f in integrated luminosity gives a factor of f' improvement in the sensitivity given by:

$$f' = f\frac{1+a}{1+\frac{a}{\sqrt{f}}} \tag{87}$$

where

$$a = \frac{\Delta B_0}{N_s + \Delta S_0}, \tag{88}$$

and ΔB_0 is the uncertainty in the background, N_s is the number of signal events assumed for Poisson statistics and ΔS_0 the systematic uncertainty in the background. For this calculation, it is assumed that there is in actual fact no background and no signal is observed.

A future computation would involve consideration of the cases with some number of signal and background events predicted.